Fractional Linear (Möbius) Transformations

The Complex Projective Line, $PSL(2,\mathbb{C})$, and Preservation of Circles/Lines

Lecture Notes

October 20, 2026

1 The Complex Projective Line

Consider $\mathbb{PC} := (\mathbb{C}^2 \setminus \{0\})/\sim$, where $\binom{z}{w} \sim \binom{z'}{w'}$ if $zw' = z'w$ – i.e. if $\binom{z}{w}$ and $\binom{z'}{w'}$ lie on the same complex line through the origin.

Natural correspondence $\widehat{\mathbb{C}} \leftrightarrow \mathbb{PC}$: a finite $\lambda \in \mathbb{C}$ corresponds to $[\binom{\lambda}{1}]$, and ∞ corresponds to $[\binom{1}{0}]$.

Setup

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible matrix, acting on $\mathbb{C}^2$ by $\binom{z}{w} \mapsto \binom{az+bw}{cz+dw}$. Since A is linear, it sends lines through the origin to lines through the origin, so it induces a well-defined map of $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, given by $z \mapsto \dfrac{az+b}{cz+d}$. This map is denoted T_A.

Notation. $GL(2,\mathbb{C}) := \{2 \times 2 \text{ matrices}, \det \neq 0\}$; $SL(2,\mathbb{C}) := \{2 \times 2 \text{ matrices}, \det = 1\}$. Every $A \in GL(2,\mathbb{C})$ satisfies $\frac{1}{\sqrt{\det A}} A \in SL(2,\mathbb{C})$.

Observe. $T_{AB}z = [AB\binom{z}{1}] = [A(B\binom{z}{1})] = T_A(T_B z)$, so $A \mapsto T_A$ is a group homomorphism, and $T_A^{-1} = T_{A^{-1}}$. Its kernel is $\{\pm I\}$ within $SL(2,\mathbb{C})$, or $\{\lambda I : \lambda \in \mathbb{C} \setminus \{0\}\}$ within $GL(2,\mathbb{C})$. *(For readers familiar with groups:*

$$PSL(2,\mathbb{C}) := SL(2,\mathbb{C})/\{\pm I\} \;\cong\; GL(2,\mathbb{C})/\{\lambda I\},$$

the group of Möbius maps.)

2 Möbius Maps Preserve Circles and Lines

Theorem

Every Möbius map sends lines and circles to lines and circles.

Proof. *Case* $c = 0$. Then $ad \neq 0$, so $a/d \neq 0$, and

$$\frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$$

is a composition of multiplication by $a/d \neq 0$ (a rotation-dilation) and a shift by b/d (a translation) – both manifestly preserve the class of lines and circles.

Case $c \neq 0$. Write

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)},$$

a composition of: a shift by d/c; the inversion $z \mapsto 1/z$; multiplication by $(bc-ad)/c^2$; and a further shift by a/c.

The map $z \mapsto 1/z$ preserves circles and lines because stereographic projection P does, and because $z \mapsto 1/z$ equals $P \circ R \circ P^{-1}$, where $R(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$ is a rotation of the sphere (a 180° rotation about the x_1-axis; note $\det R = +1$) (which trivially preserves the circles cut out on the sphere by planes, hence $P \circ R \circ P^{-1}$ preserves the circles and lines these correspond to in the plane). So the whole composition preserves circles and lines. □

Summary

Every Möbius map is a composition of **dilations** ($z \mapsto rz$, $r > 0$), **rotations** ($z \mapsto e^{i\theta} z$), **translations** ($z \mapsto z + a$), and **inversions** ($z \mapsto 1/z$).

Interactive illustration. The original notes link to a GeoGebra demonstration of this decomposition (dilation/rotation/inversion sliders, shown alongside the corresponding sphere picture): `https://www.geogebra.org/m/GhaSJw3t`.

Remark. If $a, b, c, d \in \mathbb{R}$, then $w = \dfrac{az+b}{cz+d}$ is a Möbius map with real coefficients; varying the four (real) parameters continuously deforms the map, and one can watch a triangle or square get carried by w into a curvilinear shape bounded by circular arcs – a useful way to build intuition for how far these maps can distort a region, while always sending circles/lines to circles/lines. *(Original animation: https://www.geogebra.org/m/h9MqmU3J.)*